Mathematics 216 - Differential Equations - Vahe Karamian

Antidirivative:

$\int f(x)\,dx := F(x) + C$ Where C is a constant, and the formula is from a family of antidirivitives.

$f(x) := 2x \qquad \int f(x)\,dx := 2\cdot\int x\,dx = x^2 + C$

Indefinite Integrals:

$\int x^n\,dx := \frac{x^{n+1}}{n+1} + C$ where $n \neq 1$

$\int x^{-1}\,dx := \int \frac{1}{x}\,dx = \ln(x) + C$

Difinite Integrals:

$\int_a^b f(x)\,dx := F(x)\Big|_a^b = F(b) - F(a) \qquad \lim_{n \to \infty}\left(\sum_{i=1}^{n} f(x_i)\cdot\Delta x_i\right)$

Integration Rules:
1. Substitution
2. Integration by Parts
3. Trigonometry Substitution

Example:

$\int (3\cdot x + 1)^5\,dx$ let u = 3x+1
du = 3dx $\quad dx := \frac{du}{3}$

$\int \frac{u^5}{3}\,du := \frac{1}{3}\cdot\int u^2\,du = \frac{1}{3}\cdot\frac{u^6}{6} + C := \frac{1}{18}\cdot(3\cdot x + 1)^6 + C$

Example:

$$\int u\,dv := uv - \int v\,du \qquad \int x\cdot\ln(x)\,dx$$

let u=ln(x)   dv=xdx

$$du := \frac{1}{x}\cdot dx \qquad v := \frac{x^2}{2}$$

$$\frac{x^2\cdot\ln(x)}{2} - \int \frac{x^2}{2}\cdot\frac{1}{x}\,dx := \frac{x^2\cdot\ln(x)}{2} - \frac{1}{2}\cdot\int x\,dx = \frac{x^2\cdot\ln(x)}{2} - \frac{1}{2}\cdot\frac{x^2}{2} + C := \frac{x^2\cdot\ln(x)}{2} - \frac{x^2}{4} + C$$

Integration by Parts:

$$\int f(x)\cdot g(x)\,dx = \text{first}\cdot\left(\int \text{second}\,ds\right) - \int \left(\frac{d}{dx}\text{first}\cdot\int \text{second}\,ds\right)dx$$

Example:

$$\int \ln(x)\,dx$$

let u=ln(x)   dv=dx

$$du := \frac{1}{-}\,dx \qquad v := x$$

$$uv - \int v\,du \qquad x\cdot\ln(x) - \int x\cdot\frac{1}{x}\,dx := x\cdot\ln(x) - \int 1\,dx$$

$$= \quad x\cdot\ln(x) - x + C := x\cdot(\ln(x) - 1) + C$$

Trigonometry Substitution:

$$\int \frac{1}{x^2+1}\,dx := \tan^{-1} x + C$$

Ordinary Differential Equations (ODE) - functions of single variable
Partial Differential Equations (PDE) - functions of multi-variable

1.1 Def: A Defferential Equation (DE) is an equation that has one or more derivatives of an unknown function f(x)

Ex: $\frac{dy}{dx} + x := 2$ another way to write this is y'+x=2 by I will work with $\frac{dy}{dx}$ instead of y'

$$\frac{dy}{dx} + x - y := 3 \qquad \frac{dy}{dx} + 3\cdot y := 1 \qquad \frac{d^2y}{dx^2} - 2\cdot\frac{dy}{dx} + 3\cdot y := 0$$

$$\frac{d^2y}{dx^2} + 3\cdot\frac{dy}{dx} + 5\cdot y := 1 \qquad \frac{d^3y}{dx^3} - 5\cdot y := 6\cdot x$$

Applications: Physical Sciences, Engineering

First Order Differential Equations (DE)

Methods of separation of variables

1. Separate the variables & separate the differentials (dx & dy)
2. Integrate
3. Write the solution as y=f(x)

Ex: $\frac{dy}{dx} := 3 \cdot x \quad = \quad dy := 3 \cdot x dx$ separable

$\frac{dy}{dx} := 3 \cdot x \cdot y \quad = \quad \frac{1}{y} dy := 3 \cdot x dx$ separable

Example: Solve Differential Equation y'=x

$$\frac{dy}{dx} := x \qquad dy := x \cdot dx$$

$$\int 1 \, dy := \int x \, dx$$

$$y := \frac{x^2}{2} + C$$ general solution

Example: Solve Differential Equation y'=2xy

$$\frac{dy}{dx} := 2 \cdot x \cdot y \qquad \frac{1}{y} dy := 2 \cdot x \cdot dx \qquad \int \frac{1}{y} dy := \int 2 \cdot x \, dx$$

$$\ln(y) := \frac{2 \cdot x^2}{2} + C$$

$$e^{\ln(y)} := e^{x^2 + C}$$

$$y := e^{x^2 + C} \quad = \quad e^{x^2} \cdot e^{C}$$ where C1 = e^{C}

$$y := C1 \cdot e^{x^2}$$ general solution

Example: Solve Differential Equation xy'=y^2

$$x \cdot \frac{dy}{dx} := y^2 \quad = \quad \frac{1}{y^2} dy := \frac{1}{x} dx \quad = \quad \int \frac{1}{y^2} dy := \int \frac{1}{x} dx$$

$$= \int y^{-2} dy := \int \frac{1}{x} dx \quad = \quad -\left(\frac{1}{y}\right) := \ln(x) + C$$

$$= \quad y := \frac{1}{\ln(x) + C}$$

Example: Solve Differential Equation yy'=x

$$y\cdot\frac{dy}{dx} := x \quad = \quad y\cdot dy := x\cdot dx \quad = \quad \int y\,dy := \int x\,dx$$

$$= \quad \frac{y^2}{2} := \frac{x^2}{2} + C \quad = \quad y^2 := x^2 + 2\cdot C$$

$$y := \sqrt{x^2 + 2\,C}$$

In general, a Differential Equation is separable if it can be written in the form f(y)dy=g(x)dx. It is important to recognize wheather the equation is separable or not.

Example: y'=2x+y is not separable

Example: is xy' = y^2 + 1 separable?

$$x\cdot\frac{dy}{dx} := y^2 + 1 \quad = \quad x\cdot dy := \left(y^2 + 1\right)\cdot dx \quad = \quad \frac{1}{\left(y^2+1\right)}\,dy := \frac{1}{x}\,dx$$

$$= \quad \int \frac{1}{\left(y^2+1\right)}\,dy := \int \frac{1}{x}\,dx \quad = \quad \tan^{-1} y := \ln(x) + \ln(C)$$

$$= \quad y := \tan(\ln(x\cdot C))$$

Example:

$$y\cdot\frac{dy}{dt} := e^{y^2 - 2\cdot t} \quad = \quad y\cdot\frac{dy}{dt} := e^{y^2}\cdot e^{-2\,t} \quad = \quad y\cdot dy := e^{y^2}\cdot e^{-2\cdot t}\cdot dt \quad = \quad \frac{y}{e^{y^2}}\,dy := e^{-2\cdot t}\cdot dt$$

$$= \quad \int \frac{y}{e^{y^2}}\,dy := \int e^{-2\cdot t}\,dt \quad = \quad \int y\cdot e^{-y^2}\,dy := \int e^{-2\cdot t}\,dt$$

let u = y^2
du = 2y dy
du/2 = y dy

$$\frac{1}{2}\cdot\int e^{-u}\,du := \int e^{-2\cdot t}\,dt$$

$$\frac{1}{2}\cdot\frac{e^{-u}}{-1} := \frac{e^{-2\cdot t}}{-2} + C \quad = \quad \frac{e^{-u}}{-2} := \frac{e^{-2\cdot t}}{-2} + C \quad = \quad e^{-u} := e^{-2\cdot t} - 2\cdot C$$

$$\ln\left(e^{-u}\right) := \ln\left(e^{-2\cdot t} - 2\cdot C\right) \quad = \quad -u := \ln\left(e^{-2\cdot t} - 2\cdot C\right)$$

$$-y^2 := \ln\left(e^{-2\cdot t} - 2\cdot C\right) \quad = \quad y^2 := -\ln\left(e^{-2\cdot t} - 2\cdot C\right)$$

$$y^2 := \frac{1}{\ln\left(e^{-2\cdot t - 2\cdot C}\right)}$$ we can go one more step, but it is use less

$$y := \sqrt{\frac{1}{\ln\left(e^{-2\cdot t} - 2\cdot C\right)}}$$

1.1 Testing for separability

$\frac{dy}{dx} := F(x,y)$ where F(x,y) is given in implicit format

side note:

$y^2 - 2\cdot x\cdot y + 5\cdot x + 3\cdot x\cdot y^2 := 3$ implicit format

$y := 3\cdot x^2 - 2\cdot x + 5$ explicit format

To determine wheather F(x,y) is separable or not {-> F(x,y) = f(x)*g(x)},
1. Select an x-value say x=xo and y-value say y=yo such that F(xo,yo) doesn't equal zero.
2. Determine wheather F(xo,yo)*F(x,y) = F(xo,y)*(F(x,yo)
3. If (2) is true, then

$$F(x,y) := \frac{F(x_o, y)\cdot F(x, y_o)}{F(x_o, y_o)}$$

4. If (2) is false, then F(x,y) can not be factored.

Example:

$$\frac{dy}{dx} := x^2\cdot y^2 + x^2 + y^2 + 1$$

let x = xo = 0 & y= yo = 0

$$F(x_o, y_o) := F(0,0) \quad = \quad 1 \neq 0$$

$$F(x_o, y_o)\cdot F(x,y) := F(x_o, y)\cdot F(x, y_o) \quad = \quad 1\cdot F(x,y) := (y^2+1)\cdot(x^2+1)$$

$$F(x,y) := (y^2+1)\cdot(x^2+1)$$

so we have

$$\frac{dy}{dx} := (y^2+1)\cdot(x^2+1) \quad = \quad \frac{1}{y^2+1}\,dy := (x^2+1)\cdot dx$$

$$\int \frac{1}{y^2+1}\,dy := \int (x^2+1)\,dx \quad = \quad \tan^{-1} y := \frac{x^3}{3} + x + C$$

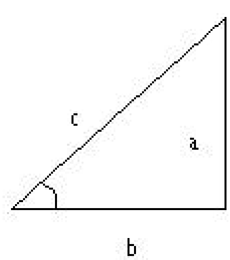
and we have $y := \tan\left(\frac{x^3}{3} + x + C\right)$

side note:

$\sin\theta := \frac{a}{c}$ $\csc\theta := \frac{c}{a}$

$\cos\theta := \frac{b}{c}$ $\cot\theta := \frac{c}{b}$

$\tan\theta := \frac{c}{b}$ $\sec\theta := \frac{c}{b}$



Go to the next page for another nice example!

Example:

$\frac{dy}{dx} := \cos(x + y) + \sin x \cdot \sin y$ since cos(x+y) = cosxcosy - sinxsiny we get the following

$\frac{dy}{dx} := \cos x \cdot \cos y - \sin x \cdot \sin y + \sin x \cdot \sin y \quad = \quad \cos x \cdot \cos y$

$\frac{dy}{dx} := \cos x \cdot \cos y \quad = \quad \frac{1}{\cos y} dy := \cos x \cdot dx$

$\int \frac{1}{\cos y} dy := \int \cos x \, dx \quad = \quad \int \sec y \, dy := \int \cos x \, dx$

$\ln(\sec y + \tan y) := \sin x + C$

$e^{\ln(\sec y + \tan y)} := e^{\sin x + C} \quad = \quad e^{\sin x} \cdot e^{C} \quad = \quad C1 \cdot e^{\sin x}$ where C1 is e^{C}

$\left| \sec y + \tan y \right| := C1 \cdot e^{\sin x}$ and you may want to solve that for y, but I wouldn't recommend!

Example:

$y \cdot \frac{dy}{dx} + 2 := e^{x} + \frac{dy}{dx} \quad = \quad y \cdot \frac{dy}{dx} - \frac{dy}{dx} := e^{x} - 2 \quad = \quad \frac{dy}{dx}(y - 1) := e^{x} - 2$

$(y - 1) \cdot dy := \left(e^{x} - 2\right) \cdot dx$

$\int (y - 1) \, dy := \int \left(e^{x} - 2\right) dx$

$\frac{y^2}{2} - y := e^{x} - 2 \cdot x + C \qquad y^2 - 2 \cdot y := 2 \cdot e^{x} - 4 \cdot x + 2 \cdot C$

$y^2 - 2 \cdot y + 1 := 2 \cdot e^{x} - 4 \cdot x + 2 \cdot C + 1 \quad = \quad 2 \cdot e^{x} - 4 \cdot x + C1$ where C1 is 2C+1

$(y - 1)^2 := 2 \cdot e^{x} - 4 \cdot x + C1$

$y - 1 := \sqrt{2 \cdot e^{x} - 4 \cdot x + C1}$

$y := 1 + \sqrt{2 \cdot e^{x} - 4 \cdot x + C1}$

$y := 1 - \sqrt{2 \cdot e^{x} - 4 \cdot x + C1}$

1.2 Applications Growth and Motion Models

Growth Model:

Example: Suppose that a population growth rate is directly proportional to the population

P' = rP -> P'/P = r = constant P = P(t)

$$\frac{dP}{dt} := r \cdot P \quad = \quad \frac{1}{P}\,dP := r \cdot dt \quad = \quad \int \frac{1}{P}\,dP := \int r\,dt$$

$$= \quad \ln(P) := r \cdot t + C$$

$$= \quad e^{\ln(P)} := e^{r \cdot t + C} \quad = \quad e^{r \cdot t} \cdot e^{C} \quad = \quad C1 \cdot e^{r \cdot t}$$ where C1 is e^{C}

$$= \quad P := C1 \cdot e^{r \cdot t}$$

$P := C1 \cdot e^{r \cdot t}$ exponential growth (can be used for bacteria, other population, continuous compund)

side notes:
y is directly proportional to x, that is y = k*x or y/x = k, where k is the constant of proportionality
y is inversly proportional to x, that is y = k/x or y*x = k
y is jointly proportional to x and z, that is y = k*x*z, or y/(x*z) = k
y is direcly proportional to x and inversly proportional to x, that is y = k*x/z or y*z/x = k

Decay Model:

Example: Let the mass of a radioactive element decay at a rate that is proportional to M(t)

$$\frac{dM}{dt} := -k \cdot M \quad = \quad \frac{1}{M}\,dM := -k \cdot dt \quad = \quad \int \frac{1}{M}\,dM := \int -k\,dt$$

$$= \quad \ln(M) := -k \cdot t + C$$

$$= \quad e^{\ln(M)} := e^{-k \cdot t + C} \quad = \quad e^{-k \cdot t} \cdot e^{C} \quad = \quad C1 \cdot e^{-k \cdot t}$$ where C1 is e^{C}

$$= \quad M := C1 \cdot e^{-k \cdot t + C}$$

$M := C1 \cdot e^{-k \cdot t}$ exponential decay model
Also known as radioactive element Half-Life Problem

Half-Life Problem: How long it takes for the initial value of a radioactive element to reduce to its half.

In General:

$\frac{dy}{dt} := c \cdot y$ which is a differential equation

if C>0, it is the growth model
if C<0, it is the decay model

Initial Value Problem:

1. at time t=0, P(t=0)=Po at t=0 $P_o := C \cdot e^0$ = $C := P_o$

Therefore: $P := P_o \cdot e^{r \cdot t}$

2. similarly at time t=0, M(t=0)=Mo at t=0 $M_o := C \cdot e^0$ = $C := M_o$

Therefore: $M := M_o \cdot e^{-k \cdot t}$

Doubling Time, Tripling Time $P := C \cdot e^{r \cdot t}$ continues comound interest

Doubling Time Formula: How long will it take to double the original investment?

$P := P_o \cdot e^{r \cdot t}$

$2 \cdot P_o := P_o \cdot e^{r \cdot t}$ = $2 := e^{r \cdot t}$ = $\ln(2) := \ln\left(e^{r \cdot t}\right)$ = $\ln(2) := r \cdot t$

$t := \frac{\ln(2)}{r}$ = $\frac{\ln(m)}{r}$

$t := \frac{\ln(m)}{r}$ General formula

Same goes for decay half-life:

$M := M_o \cdot e^{-k \cdot t}$

$\frac{1}{2} \cdot M_o := M_o \cdot e^{-k \cdot t}$ = $\frac{1}{2} := e^{-k \cdot t}$ = $\ln\left(\frac{1}{2}\right) := \ln\left(e^{-k \cdot t}\right)$ = $\ln\left(\frac{1}{2}\right) := -k \cdot t$

$t := \frac{\ln\left(\frac{1}{2}\right)}{-k}$ = $\frac{\ln(1) - \ln(2)}{-k} := \frac{\ln(2)}{k}$

$t := \frac{\ln(2)}{k}$ General formula

On the next page I will cover Logistic Growth

Logistic Growth: (Logistic Growth is due to a limiting factor) - Limited Growth

Model: $\frac{dy}{dt} := a \cdot y + b \cdot y^2$ where a, b doesn't equal 0

$$\frac{dy}{dt} := a \cdot y + b \cdot y^2 \quad = \quad \frac{1}{a \cdot y + b \cdot y^2} \cdot dy := dt$$

$$= \int \frac{1}{a \cdot y + b \cdot y^2} dy := \int 1 \, dt$$

Side Note: This section for the integral demonstration!!!

$$\frac{1}{x \cdot (2 + 3 \cdot x)} := \frac{A}{x} + \frac{B}{(2 + 3 \cdot x)}$$

$$= \frac{1}{x \cdot (2 + 3 \cdot x)} := \frac{A \cdot (2 + 3 \cdot x) + B \cdot x}{x \cdot (2 + 3 \cdot x)}$$

$$= 1 := A \cdot (2 + 3 \cdot x) + B \cdot x$$

$$= 1 := 2 \cdot A + 3 \cdot A \cdot x + B \cdot x$$

$$= 1 := x \cdot (3 \cdot A + B) + 2 \cdot A$$

$$3 \cdot A + B := 0 \quad = \quad B := -3 \cdot A \quad B := -3 \cdot \frac{1}{2} \quad B = -1.5$$

$$2 \cdot A := 1 \quad = \quad A := \frac{1}{2}$$

$$= \int \frac{1}{y \cdot (a + b \cdot y)} dy := \int 1 \, dt$$

$$= \int \frac{1}{a} \cdot \left(\frac{1}{y} - \frac{b}{a \cdot b \cdot y} \right) dy := \int 1 \, dt$$

$$= \ln(y) - \ln(a + b \cdot y) := a \cdot t + C$$

$$= \ln\left(\frac{y}{a + b \cdot y} \right) := a \cdot t + C$$

$$= e^{\ln\left(\frac{y}{a + b \cdot y}\right)} := e^{a \cdot t + C} \quad = \quad \frac{y}{a + b \cdot y} := e^{C} \cdot e^{a \cdot t} \quad = \quad C1 \cdot e^{a \cdot t}$$ where $C1 := e^{C}$

$$= \frac{a + b \cdot y}{y} := \frac{1}{C1} \cdot e^{-a \cdot t} \quad = \quad \frac{a}{y} + b := C2 \cdot e^{-a \cdot t}$$ where $C2 := \frac{1}{C1}$

$$= \frac{a}{y} := C2 \cdot e^{-a \cdot t} - b$$

$$= y := \frac{a}{C2 \cdot e^{-a \cdot t} - b}$$

Example:

$$\frac{dQ}{dt} := \frac{-10}{100 - t} \cdot Q(t) \quad = \quad \int \frac{1}{Q}\, dQ := \int \frac{-10}{100 - t}\, dt$$

$$= \quad \ln(Q) := 10 \cdot \int \frac{1}{t - 100}\, dt \quad = \quad 10 \cdot \ln(t - 100) + \ln(C)$$

$$= \quad Q := (100 - t)^{10} \cdot C$$

$$Q(0) := 20 \quad = \quad (100 - t)^{10} \cdot C := (100 - 0)^{10} \cdot C \quad = \quad 100^{10} \cdot C$$

$$C := \frac{20}{100^{10}}$$

$$Q := (100 - t)^{10} \cdot \frac{20}{100^{10}} \quad = \quad 20 \cdot \left(\frac{100 - t}{100}\right)^{10}$$

$$Q(t) := 20 \cdot \left(1 - \frac{t}{100}\right)^{10}$$

Lets move on to linear and rational substitutions!

Section 1.3 Linear and Rational Substitutions

Goal: change from non-separable differential equation to separable differential equation

1. Linear Substitution:

$u := a\cdot x + b\cdot y + c$

Ex:

$\frac{dy}{dx} := a\cdot x + b\cdot y + c$ where a,b, and c are constants

notice that x is the independant variable and y is replaced by the new variable 'u' (temporarly)

2. Rational Substitution:

$u := \frac{y}{x}$ where x is the independant variable and y is replaced by the new variable 'u' (temporarly)

Theorem:

If the given D.E. (Differential Equation) can be written as $\frac{dy}{dx} := F(u)$, where u = ax + by + c, or u = y/x, and F(u) is any function or 'u', then the D.E. is separable, when it is written in terms of x & u.

Reason: Linear

$\frac{dy}{dx} := a\cdot x + b\cdot y + c$ let u = ax + by + c

$\frac{dy}{dx} := u$ or F(u) $\frac{dy}{dx} := F(u)$

$u := a\cdot x + b\cdot y + c$

$du := a\cdot dx + b\cdot dy + 0$

$\frac{du}{dx} := a\cdot\frac{dx}{dx} + b\cdot\frac{dy}{dx}$ = $\frac{du}{dx} := a + b\cdot\frac{dy}{dx}$

$\frac{dy}{dx} := \frac{1}{b}\cdot\left(\frac{du}{dx} - a\right)$

$\frac{1}{b}\cdot\left(\frac{du}{dx} - a\right) := F(u)$

$\frac{du}{dx} - a := b\cdot F(u)$

$\frac{du}{dx} := b\cdot F(u) + a$

$\int \frac{1}{b\cdot F(u) + a}\,du := \int 1\,dx$ ...

Reason: Rational

if u = y/x, than y = u*x $\quad \frac{dy}{dx} := u + x\cdot\frac{du}{dx}$

$\frac{dy}{dx} := F(u) \qquad F(u) := u + x\cdot\frac{du}{dx} \qquad u\cdot\frac{du}{dx} := F(u) - u \qquad \frac{du}{F(u) - u} := x\cdot dx$

$$\int \frac{1}{F(u) - u}\,du := \int \frac{1}{x}\,dx$$

Example: $\frac{dy}{dx} := 1 - \frac{y}{x} - \frac{y^2}{x^2} \quad = \quad 1 - \left(\frac{y}{x}\right) - \left(\frac{y}{x}\right)^2$ $\qquad$ let u = y/x

$y := u\cdot x$

$\frac{dy}{dx} := u + x\cdot\frac{du}{dx}$

$$u + x\cdot\frac{du}{dx} := 1 - u - u^2$$

$$x\cdot\frac{du}{dx} := 1 - 2\cdot u - u^2$$

$$\frac{1}{1 - 2\cdot u - u^2}\cdot du := \frac{1}{x}\cdot dx$$

$$\int \frac{1}{1 - 2\cdot u - u^2}\,du := \int \frac{1}{x}\,dx$$

$$\int \frac{1}{(u+1)^2 - 2}\,du := \int \frac{1}{x}\,dx$$

let t = u + 1

$$-\int \frac{1}{t^2 - \left(\sqrt{2}\right)^2}\,dt := \int \frac{1}{x}\,dx$$

$$\frac{1}{2\left(\sqrt{2}\right)}\cdot\left(\ln\left(t - \sqrt{2}\right) - \ln\left(t + \sqrt{2}\right)\right) := \ln(x) + \ln(C)$$

$$\frac{1}{2\left(\sqrt{2}\right)}\cdot\left[\ln\left[(u+1) - \sqrt{2}\right] - \ln\left[(u+1) + \sqrt{2}\right]\right] := \ln(x) + \ln(C)$$

$$\frac{1}{2\left(\sqrt{2}\right)}\cdot\left[\ln\left[\left(\frac{y}{x} + 1\right) - \sqrt{2}\right] - \ln\left[\left(\frac{y}{x} + 1\right) + \sqrt{2}\right]\right] := \ln(x) + \ln(C)$$

How to recognize when to use rational substitution:

For an equation in the form F(x,y)dy = G(x,y)dx where F(x,y) and G(x,y) are polynomials in x & y, suppose each term on both polynomials has the same degree, then rational substitution might be appropriate.

A Differential Equation of the type $\frac{dy}{dx} := F\left(\frac{a_1 \cdot x + a_2 \cdot y + a_3}{b_1 \cdot y + b_2 \cdot y + b_3}\right)$ can be solved by linear substitution

or by changing variables x = t + A and y = u + B, so that a rational substitution applies.

Example:

$\frac{dy}{dx} := \frac{-x + y}{x + y + 1}$

let u = x+y

$1 + \frac{dy}{dx} := \frac{du}{dx} \quad = \quad \frac{du}{dx} - 1 := \frac{dy}{dx}$

$\frac{du}{dx} - 1 := \frac{-u}{u + 1}$

$\frac{du}{dx} := 1 - \frac{u}{u + 1} \quad = \quad \frac{1}{u + 1}$

$(u + 1) \cdot du := dx$

$\int (u + 1)\, du := \int 1\, dx$

$\frac{u^2}{2} + u := x + C$

$u^2 + 2 \cdot u := 2 \cdot x + 2 \cdot C$

$(x + y)^2 + 2 \cdot (x + y) := 2 \cdot x + 2 \cdot C$

Example:

$\frac{dy}{dx} := \sin^2(x - y)$

let u = x - y

$\frac{du}{dx} := 1 - \frac{dy}{dx}$

$\frac{dy}{dx} := 1 - \frac{du}{dx}$

$1 - \frac{du}{dx} := \sin^2(u)$

$\frac{du}{dx} := 1 - \sin^2 u \quad = \quad \cos^2 u := \frac{1 + \cos(2 \cdot u)}{2}$

$dx := \frac{2}{1 + \cos(2 \cdot u)} \cdot du$

$\int 1\, dx := \int \sec^2 u\, du$

$= \quad C + x := \tan(u)$

$= \quad u := \tan^{-1}(x + C)$

$= \quad x - y := \tan^{-1}(x + C)$

$= \quad y := x - \tan^{-1}(x + C)$ General Formula

Example:

$$(x-2\cdot y)\cdot dy := (4\cdot x - y)\cdot dx$$

we divide by x and obtain the following:

$$\frac{x-2\cdot y}{x}\cdot dy := \frac{4\cdot x - y}{x}\cdot dx$$

$$1-2\cdot\left(\frac{y}{x}\right)\cdot dy := 4-\frac{y}{x}\cdot dx$$

let u = y/x

$$y := u\cdot x$$

$$\frac{dy}{dx} := u + x\cdot\frac{du}{dx}$$

$$\frac{dy}{dx} := \left(\frac{4-u}{1-2\cdot u}\right)$$

$$u + x\cdot\frac{du}{dx} := \left(\frac{4-u}{1-2\cdot u}\right) \quad = \quad x\cdot\frac{du}{dx} := \left(\frac{4-u}{1-2\cdot u} - u\right) \quad = \quad x\cdot\frac{du}{dx} := \frac{4-uu4+2\cdot u^2}{1-2\cdot u}$$

$$x\cdot\frac{du}{dx} := \frac{2\cdot u^2 - 2\cdot u - 4}{1-2\cdot u} \quad = \quad \int \frac{1-2\cdot u}{2\cdot u^2 - 2\cdot u - 4}\,du := \int \frac{1}{x}\,dx \quad = \quad \frac{-1}{2}\cdot\int \frac{2\cdot u - 1}{u^2 - u + 2}\,du := \int \frac{1}{x}\,dx$$

let $t := u^2 - u + 2$

$$dt := (2\cdot u - 1)\cdot du \qquad \frac{-1}{2}\cdot\int \frac{1}{t}\,dt := \int \frac{1}{x}\,dx \quad = \quad \frac{-1}{2}\cdot\ln(t) := \ln(x) + \ln(C)$$

$$= \quad \frac{-1}{2}\cdot\ln\left(u^2 - u + 2\right) := \ln(C\cdot x)$$

$$= \quad \ln\left(u^2 - u + 2\right) := -2\cdot\ln(C\cdot x)$$

$$= \quad e^{\ln\left(\left(u^2-u+2\right)\right)} := e^{-2\cdot\ln(C\cdot x)}$$

$$= \quad u^2 - u + 2 := \frac{1}{C^2\cdot x^2}$$

$$= \quad \left(\frac{y}{x}\right)^2 - \left(\frac{y}{x}\right) - 2 := \frac{1}{C^2\cdot x^2}$$ General Formula

Example:

$$\frac{\frac{d}{dA}(t)}{A(t)\cdot(N-A(t))} := k \quad = \quad \int \frac{1}{A(t)\cdot(N-A(t))}\,dA := k\cdot\int 1\,dt$$

$$\int \left(\frac{1}{A(t)} + \frac{1}{N-A(t)}\right) d\frac{1}{N}\cdot A := k\cdot\int 1\,dt \quad = \quad \int \frac{1}{A(t)}\,dA - \int \frac{1}{-N+A(t)}\,dA := N\cdot k\cdot\int 1\,dt$$

Continued on the next page

Example: (Continued)

$$\ln(A) - \ln(A - N) := N \cdot k \cdot t + C$$

$$\ln\left(\frac{A}{A - N}\right) := N \cdot k \cdot t + C \quad = \quad e^{\ln\left(\frac{A}{A-N}\right)} := e^{N \cdot k \cdot t + C} \quad = \quad \frac{A}{A - N} := e^{N \cdot k \cdot t + C} \quad = \quad C_1 \cdot e^{N \cdot k \cdot t}$$

$$\frac{A - N}{A} := \frac{1}{C_1} \cdot e^{-N \cdot k \cdot t} \quad = \quad 1 - \frac{N}{A} := C_2 \cdot e^{-N \cdot k \cdot t} \quad \text{where } C_2 := \frac{1}{C_1}$$

$$\frac{N}{A} := 1 - C_2 \cdot e^{-N \cdot k \cdot t} \quad = \quad A := \frac{N}{1 - C_2 \cdot e^{-N \cdot k \cdot t}}$$