Three-Dimensional Analytic Geometry and Vectors

Section 11.1 Three-Dimensional Coordinate Systems

Distance formula in three dimensions
The distance |P1P2| between the points P1(x1,y1,z1) and P2(x2,y2,z2) is

$$|P1P2| := \sqrt{(x2 - x1)^2 + (y2 - y1)^2 + (z2 - z1)^2}$$

Equation of a shpere
An equation of a sphere with center C(h,k,l) and radius r is

$$r^2 := (x - h)^2 + (y - k)^2 + (z - l)^2$$

In particular, if the center is the origion O, then an equation of the sphere is

$$r^2 := x^2 + y^2 + z^2$$

Section 11.2 Vectors

Given the points A(x1,y1,z1) and B(x2,y2,z2), the vector **a** with representation AB is
a = <x2-x1, y2-y1, z2-z1>

The length of the three-dimensional vector **a** = <a1,a2,a3> is $|a| := \sqrt{a1^2 + a2^2 + a3^2}$

Vector Addition
if **a** = <a1,a2> and **b** = <b1,b2>, then the vector **a+b** is defined by **a+b** =<a1+b1, a2+b2>
similarly, for three-dimensional vectors, <a1,a2,a3>+<b1,b2,b3> = <a1+b1, a2+b2, a3+b3>

Multiplication of a vector by a scalar
If c is **a** scalar and **a** = <a1,a2>, then the vector c**a** is defined by c**a** = <ca1, ca2>
similarly, for three-dimensional vectors, c<a1,a2,a3> = <ca1,ca2,ca3>

Section 11.3 The Dot Product

Definition
If **a** = <a1,a2,a3> and **b** = <b1,b2,b3>, then the dot product of **a** and **b** in the number **a*b** given by

$$a \cdot b := a1 \cdot b1 + a2 \cdot b2 + a3 \cdot b3$$

Theorem
If θ is the angle between the vectors **a** and **b**, then $a \cdot b := |a| \cdot |b| \cdot \cos(\theta)$

If θ is the angle between the nonzero vectors **a** and **b**, then $\cos(\theta) := \dfrac{a \cdot b}{|a| \cdot |b|}$

a and **b** are orthogonal if and only if $a \cdot b := 0$

Scalar projection of **b** onto **a**: $b := \dfrac{a \cdot b}{|a|}$

Vector projection of **b** onto **a**: $b := \left(\dfrac{a \cdot b}{|a|}\right) \cdot \dfrac{a}{|a|}$ which is $\dfrac{a \cdot b}{(|a|)^2} \cdot a$

Section 11.4 The Cross Product

Definition
If **a** = <a1,a2,a3> and **b** = <b1,b2,b3>, then the cross product of **a** and **b** is the vector
a x b = <a2b3-a3b2, a3b1-a1b3, a1b2-a2b1>

Theprem
The vector **a x b** is orthogonal to both **a** and **b**

Theorem
[redacted]

Corollary
Two nonzero vectors **a** and **b** are parallel if and only if **a x b** = 0

Section 11.5 Equations of Lines and Planes

$r := ro + t \cdot v$ is the vector equation of a line L
$x := x0 + a \cdot t$
$y := y0 + b \cdot t$ are the parametric equations of the line L
$z := z0 + c \cdot t$
$\frac{x - xo}{a} = \frac{y - yo}{b} = \frac{z - zo}{c} = t$ is the symmetric equation of the line L

$n \cdot (r - ro) := 0$ is the vector equation of a plane P
To obtain a scalar equation for the plane, we write **n** = <a,b,c>, **r** = <x,y,z>, and **ro**=<xo,yo,zo>, then we obtain the following: <a,b,c>*<x-xo, y-yo, z-zo> = 0

$a \cdot (x - xo) + b(y - yo) + c \cdot (z - zo) := 0$ scalar equation of the plane through *Po(xo,yo,zo)* with normal vector **n** = <a,b,c> we can write the equation of a plane as ax+by+cz = d.

Distance D from a point P(xo,yo,zo) to the plane ax+by+cz+d = 0

$$D := \frac{|a \cdot x1 + b \cdot y1 + c \cdot z1 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

Section 11.6 Quadric Surfaces

Ellipsoids $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} := 1$ Hyperboloids $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} := 1$ Cones $\frac{z^2}{c^2} := \frac{x^2}{a^2} + \frac{y^2}{b^2}$

Paraboloids $\frac{z}{c} := \frac{x^2}{a^2} + \frac{y^2}{b^2}$ Quadric Cylinders $1 := \frac{x^2}{a^2} + \frac{y^2}{b^2}$

Section 11.7 Vector Function and Space Curves

We now study functions whose alues are vectors because such functions are needed to describe curves in space andd the motion of particles in space.

A **vector-values function**, or **vector function**, is simply a function whose domain is a set of real numbers and whose range is a set of vectors.

If **r**(t) = <f(t), g(t), h(t)>, then

$$\lim_{t \to a} r(t) = \left(\lim_{t \to a} f(t),\ \lim_{t \to a} g(t),\ \lim_{t \to a} h(t) \right)$$

provided the limits of the component functions exist.

Derivatives and Integrals

$$\frac{dr}{dt} = r(t) := \lim_{h \to 0} \frac{r(t+h) - r(t)}{h}$$

Theorem

If **r**(t) = <f(t), g(t), h(t)> = f(t)i + g(t)j + h(t)k, where f,g, and h are differentiable functions, then
r'(t) = <f'(t), g'(t), h'(t)> = f'(t)i + g'(t)j + h'(t)k

Theorem

Suppose u and v are differentiable vector functions, c is a scalar, and f is a real-valued function, then:

$$\frac{d}{dt}(u(t) + v(t)) = \frac{d}{du}(t) + \frac{d}{dv}(t)$$

$$\frac{d}{dt}(c\,u(t)) = c\,\frac{d}{du}(t)$$

$$\frac{d}{dt}(f(t)\,u(t)) = \frac{d}{df}(t)\,u(t) + f(t)\,\frac{d}{du}(t)$$

$$\frac{d}{dt}(u(t)\cdot v(t)) = \frac{d}{du}(t)\cdot v(t) + u(t)\cdot\frac{d}{dv}(t)$$

$$\frac{d}{dt}(u(t) \times v(t)) = \left[\left[\frac{d}{du}(t)\right] \times v(t)\right] + \left[u(t) \times \left[\frac{d}{dv}(t)\right]\right]$$

$$\frac{d}{dt}(u(f(t))) = \frac{d}{df}(t)\,\frac{d}{du}(f(t))$$ Chain Rule

Section 11.8 Arc Length and Curviture
Recall that we defined the length of a plane curve x = f(t), y = g(t), a<=t<=b, as the limit of lengths of inscribed polygons and, for the case where f' and g' are continuous, we arrived at the formula:

$$L = \int_a^b \sqrt{\left[f^{p}(t)\right]^2 + \left[g^{p}(t)\right]^2}\, dt = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

The length of a space curve is defined in exactly the same way. Suppose that the curve has the vector equation **r**(t)=<f(t), g(t), h(t)>, a<=t<=b, or equivalently, the parametric equations x=f(t), y=g(t), and z=h(t), where f', g', and h' are continuous. If the curve is traversed exactly once as *t* increases from *a* to *b*, then it can be shown that its length is:

$$L = \int_a^b \sqrt{\left[f^{p}(t)\right]^2 + \left[g^{p}(t)\right]^2 + \left[h^{p}(t)\right]^2}\, dt = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$$

Notice that both of the arc length formulas (1) and (2) can be put into the more compact form

$$L = \int_a^b \left| r^{p}(t) \right| dt$$

<u>Definition</u>
The **curviture** of a curve is $k = \left| \frac{dT}{ds} \right|$ where **T** is the unit tangent vector.

The curviture is easier to compute if it is expressed in terms of the parameter t instead of s, so we use the Chain Rule to write

$\frac{dT}{dt} = \frac{dT}{ds} \cdot \frac{ds}{dt}$ and $k = \left| \frac{dT}{ds} \right| = \left| \frac{\left(\frac{dT}{dt}\right)}{\left(\frac{ds}{dt}\right)} \right|$ but ds/dt = |r'(t)|, so $k(t) := \frac{\left| T^{p}(t) \right|}{\left| r^{p}(t) \right|}$

<u>Theorem</u>
The curviture of the curve given by the vector function **r** is $k(t) := \frac{\left| r^{p}(t) \times r^{pp}(t) \right|}{\left[\left| r^{p}(t) \right| \right]^3}$

$$k(x) := \frac{\left| f^{pp}(x) \right|}{\left[1 + \left[f^{p}(x) \right]^2 \right]^{\frac{2}{3}}}$$

Example 1:
Find the length of the arc to the circular helix with vector equation r(t)=cos(t)i+sin(t)j+tk from the point (1,0,0) to the point (1,0,2π).

Since r'(t) = -sin(t)i + cos(t)j + k, we have

$$|r^p(t)| = \sqrt{(-\sin(t))^2 + \cos^2(t)} = \sqrt{2}$$

The arc from (1,0,0) to (1,0,2π) is described by the parameter interval 0<=t<=2π and so we have

$$L = \int_0^{2\pi} |r^p(t)|\,dt = \int_0^{2\pi} \sqrt{2}\,dt = 1\circ 2\sqrt{2}\,\pi$$

Example 2:
Reparametrize the helix r(t)=cos(t)i+sin(t)j+tk with respect to arc length measured from (1,0,0) in the direction of increasing t.

The initial point (1,0,0) corresponds to the parameter value t = 0. From Example 1 we have

$$\frac{ds}{dt} = |r^p(t)| = \sqrt{2} \quad \text{and so} \quad s := s(t) \quad s(t) := \int_0^t |r^p(u)|\,du \quad \int_0^t \sqrt{2}\,du = \sqrt{2}\cdot t$$

Threrefore $t := \frac{s}{\sqrt{2}}$ and the required reparametrization is obtained by substituting for t:

$$r(t(s)) = \cos\left(\frac{s}{\sqrt{2}}\right) i + \sin\left(\frac{s}{\sqrt{2}}\right) j + \left(\frac{s}{\sqrt{2}}\right) k$$

Example 3:
Show that the curvature of a circle of radius a is 1/a

We can take the circle to have center the origin, and then a parametrization is

$r(t) = a\cdot\cos(t)\,i + a\cdot\sin(t)\,j$ therefore $r^p(t) = -a\cdot\sin(t)\,i + a\cdot\cos(t)\,j$ and $|r^p(t)| = a$

so $T(t) = \frac{r^p(t)}{|r^p(t)|} = -\sin(t)\,i + \cos(t)\,j$ and $T^p(t) = -\cos(t)\,i - \sin(t)\,j$

This gives $|T^p(t)| = 1$ so we have $k(t) = \frac{|T^p(t)|}{|r^p(t)|} = \frac{1}{a}$

This shows that small circles have large curvatures and large circles have small curvatures

Example 4:
Find the curvature of the twisted cubic r(t)=<t,t^2,t^3> at a general point and at (0,0,0)

We first compute the required ingredients:

r'(t)=<1,2t,3t^2. and r''(t)=<0,2,6t>

$$|r^{p}(t)| := \sqrt{1+4t^2+9t^4} \qquad r^{p}(t)\times r^{pp}(t) := \begin{bmatrix} i & j & k \\ 1 & 2t & 3t^2 \\ 0 & 2 & 6t \end{bmatrix} \quad 6\cdot t^2\, i + 6\cdot t\, j + 2\, k$$

$$|r^{p}(t)\times r^{pp}(t)| := \sqrt{36\cdot t^4+36\cdot t^2+4} \quad 2\sqrt{9\cdot t^4+9\cdot t^2+1}$$

so k(t) = $\dfrac{|r^{p}(t)\times r^{pp}(t)|}{\left[|r^{p}(t)|\right]^3} := \dfrac{2\sqrt{1+9\cdot t^2+9\cdot t^4}}{\left(1+4\cdot t^2+9\cdot t^4\right)^{\frac{3}{2}}}$ at the origin the curve is k(0) = 2

Example 5:
Find the curvature of the parabola y=x^2 at the points (0,0), (1,1), and (2,4)

sine y' = 2x and y'' = 2, we get

$$k(x) = \frac{\left|\frac{d^2}{dy^2}\right|}{\left[1+\left(\frac{d^2}{dy^2}\right)^3\right]^{\frac{2}{3}}} = \frac{2}{\left(1+4x^2\right)^{\frac{2}{3}}}$$

The curvature at (0,0) is k(0) = 2.
At (1,1) it is k(1) = 2/(5)^3/2 = 0.18.
At (2,4) it is k(2) = 2/(17)^./2 = 0.03.

Example 6:
Find the unit normal and binormal vectors for the circular helix r(t) = cos(t)i + sin(t)j + tk

$$\frac{d}{dr}(t) = -\sin(t)\ i + \cos(t)\ j + k \qquad \left|\frac{d}{dr}(t)\right| = \sqrt{2}$$

$$T(t) := \frac{\frac{d}{dr}(t)}{\left|\frac{d}{dr}(t)\right|} = \frac{1}{\sqrt{2}}(-\sin(t)\ i + \cos(t)\ j + k)$$

$$\frac{d}{dT}(t) = \frac{1}{\sqrt{2}}(-\cos(t)\ i - \sin(t)\ j) \qquad \left|\frac{d}{dT}(t)\right| = \frac{1}{\sqrt{2}}$$

$$N(t) = \frac{\frac{d}{dT}(t)}{\left|\frac{d}{dT}(t)\right|} = -\cos(t)\ i - \sin(t)\ j = (-\cos(t), -\sin(t), 0)$$

This shows that the normal vector at a point on the helix is horizontal and points toward the z-axis. The binormal vector is

$$B(t) := T(t) \times N(t) = \frac{1}{\sqrt{2}}\begin{bmatrix} i & j & k \\ -\sin(t) & \cos(t) & 1 \\ -\cos(t) & -\sin(t) & 0 \end{bmatrix} = \frac{1}{\sqrt{2}}(\sin(t), -\cos(t), 1)$$

Example 7:
Find the equation of the normal plane and osculating plane of the helix in *Example 6* at the point P(0,1,π/2).

The normal plane at P has normal vector r'(π/2)=<-1,0,1>, so an equation is

$$-1\ (x - 0) + 0\ (y - 1) + 1\left(z - \frac{\pi}{2}\right) = 0 \quad \text{or} \quad z := x + \frac{\pi}{2}$$

The osculating plane at P contains the vectors **T** and **N**, so its normal vector is **TxN=B**. From *Example 6* we have

$$B(t) := \frac{1}{\sqrt{2}}(\sin(t), -\cos(t), 1) \qquad B\left(\frac{\pi}{2}\right) = \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$$

A simpler normal vector is <1,0,1>, so an equation of the osculating plane is

$$1\ (x - 0) + 0\ (y - 1) + 1\left(z - \frac{\pi}{2}\right) = 0 \quad \text{or} \quad z := -x + \frac{\pi}{2}$$

Example 8:
Find and graph the osculating circle of the parabola y=x^2 at the origin.

From *Example 5* the curvature of the parabola at the origin is k(0)=2. So the radius of the osculating circle at the origin is 1/k = 1/2 and its center is (0,1/2). Its equation is therefore

$$x^2 + \left(y - \frac{1}{2}\right)^2 = \frac{1}{4}$$

We use parametric equations of this circle:

$$x := \frac{1}{2}\cos(t)$$

$$y := \frac{1}{2} + \frac{1}{2}\sin(t)$$

Section 11.9 Motion In Space: Velocity And Acceleration
In this section we show how the ideas of tangent and normal vectors and curvature can be used in physics to study the motion of an object, including its velocity and acceleration, along a space curve.